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Taylor diffusion in time-dependent flow

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Abstract

In this paper, we consider the problem of calculating the axial concentration profile of a solute transported by a time-dependent flow in a rigid straight pipe. This generalizes the result that Taylor derived in 1953 for calculating the axial concentration profile of a solute in a steady flow. Using asymptotic analysis, we derive a time-dependent diffusion equation for the mean concentration profile along the axial-direction in a pipe. In the special case of time-independent flow, our result reduces to that of Taylor.

Keywords: Taylor diffusion; Time-dependent flow; MRI; Asymptotic analysis; Convection-diffusion

1. Introduction

In 1953, Sir Geoffrey Taylor considered the problem of measuring the concentration of a solute in a slowly moving flow in a rigid pipe [1]. Taylor assumed a Poiseuillian velocity profile of the form $u_0(1-\frac{r^2}{a^2})$, where u_0 is the maximum velocity along the axis of the pipe, a is the radius of the pipe and r is the radial coordinate. He showed that in the frame of reference moving with velocity $\frac{u_0}{2}$, the concentration of the solute satisfied the diffusion equation, with a diffusion coefficient of $\frac{a^2u_0^2}{192D}$, where D is the coefficient of molecular diffusion for the solute. In this paper, we generalize Taylor's result to time-dependent flows and derive the corresponding diffusion equation for the mean concentration of the solute in an axisymmetric setting. Some related work includes the dispersion of solute in turbulent flow in a pipe [2], dispersion of solute in a pipe where the distribution of the solute is described in terms of its moments in the flow direction [3], studies of the dispersion coefficient of a chromatographic system where the velocity field is time periodic [4,5] and a revision of the Taylor limit where the validity of the theory is extended to better resolution and earlier times [6].

Calculating the concentration profile of an agent at a given site of action is important in certain Magnetic Resonance Imaging (MRI) studies. Consider a fast-acting drug (compared to total circulation time) that is administered in some vein. As an example, take the target area to be some region in the brain. Using our time-dependent Taylor diffusion equation and supposing we have a complete circulatory model, one can calculate the concentration profile at the site of action in the brain. Dynamic susceptibility contrast (DSC) MRI studies for imaging the brain are examples of studies where knowing the concentration profile of the administered agent at the site of action is important. In DSC MRI, a bolus injection of a paramagnetic contrast agent is administered, and its passage through the vasculature is

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monitored by serial measurement of the MR signal loss in the surrounding tissue. The scan time is typically very short, on the order of a few seconds, and the time it takes for an administered agent to get from a vein to the brain is also on the order of a few seconds. A contrast agent, such as gadolinium diethylenetriamine pentaaceticacid (Gd-DTPA), is used to introduce contrast in the MRI image. Gd-DTPA does not penetrate the blood brain barrier, but stays in the vasculature. The primary interests in a DSC MRI study is to extract the blood volume map and perfusion map for the region of interest. Since the scan time and the time taken to reach the site of action from the site of administration is short (a few seconds) compared to the time of one complete circulation (approximately one minute in humans), how the agent is diffused in the vasculature is important when calculating the concentration profile. In calculating the cerebral blood flow (CBF), the concentration profile of interest is the concentration of contrast agent entering the region of interest, and is often referred to as the arterial input function (AIF). The AIF is typically estimated from a major artery, such as the middle cerebral artery, and assumed to be the input to the tissue. However, ignoring dispersion and delay effects that take place from the site of measurement of the AIF to the tissue site can cause significant errors in the quantification of CBF [7,8]. Since DSC MRI is commonly used in clinical studies of cerebral ischemia, these errors lead to inaccurate information on stroke. Using the time-dependent Taylor diffusion equation we derive, and a circulation model, one can calculate the AIF at the site of action and therefore derive more accurate perfusion maps.

Researchers at the Toshiba Stroke Research Center and Aerospace Engineering and Radiology used an insoluble agent to improve flow measurement and quantification of arteriovenous malformation (AVM) [9]. The improvement on AVM flow measurement and visualization was achieved by using Ethiodol®, an insoluble ethiodized oil as the contrast agent, as opposed to conventionally used soluble agents in angiographic studies. The insoluble Ethiodol® has a much shorter transit time than soluble contrasts. The authors explain that a likely cause of a shorter transit time is that, according to Taylor's result, dispersion of the soluble agent is driven both by convective transport and diffusion.

Suppose a substance is present at time $t = 0$ with initial concentration $f(x)$ in a rigid straight pipe, where the velocity field is given by $V(r, t; a)$ and x is the distance along the axis of the pipe. The spread of the substance in the fluid is primarily due to the combined effect of molecular diffusion in the radial direction and convection parallel to the axis of motion. Assume that we are in a regime where variations in the concentration caused by convective transport occur in a time frame that is much longer than the time it takes for appreciable radial concentration differences to dissipate. We show that the axial concentration profile of the substance is dispersed according to a process that obeys the diffusion equation, with a certain variable diffusion coefficient, when considered in a particular frame of reference. The general time-dependent diffusion equation we derive reduces to Taylor's result in the time independent case. We then apply the resulting generalized equations to calculate the mean concentration along a pipe where the velocity field is one that comes from axisymmetric flow with a pressure gradient that is sinusoidal in time.

2. Derivation of the generalized Taylor diffusion equation

Suppose we have an infinite circular pipe described in the $r - x$ plane by $x = -\infty$ to $x = +\infty$, $r = 0$ to a, where the axial velocity field is given in cylindrical coordinates by $V(r, t; a)$, and the radial velocity is zero. Note that the axial velocity profile is independent of x. The dependence on a is made explicit so that later we can vary a in order to study the asymptotic limit that leads to Taylor diffusion. The axisymmetric convection-diffusion equation for the concentration $C(r, x, t)$ of a solute introduced into the flow is given by

$$D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right) = \frac{\partial C}{\partial t} + V(r, t; a)\frac{\partial C}{\partial x} \quad (1)$$

where

C	is the concentration of solute,
D	is the coefficient of molecular diffusion of the solute,
r	is the radial variable,
x	is distance parallel to axis of the pipe,
t	is time,

$V(r, t; a) = u_0(t)F(r, t; a)$ is the axial velocity of the moving fluid, where $\frac{u_0(t)}{2}$ is the average velocity over a cross-section at a given time t.

Let $x_1 = x - x_0(t)$, where $x_0(t)$ will be chosen later, and also let $t_1 = t$. Then define $C_\alpha\ (r, x_1, t_1) = C(r, x, t)$ at corresponding points, that is $C_\alpha(r, x - x_0(t), t) = C \times (r, x, t)$. It follows that $\frac{\partial C}{\partial x} = \frac{\partial C_\alpha}{\partial x_1}$ and $\frac{\partial C}{\partial t} = -x_0'(t)\frac{\partial C_\alpha}{\partial x_1} + \frac{\partial C_\alpha}{\partial t_1}$, where $x_0'(t)$ is the derivative with respect to time of $x_0(t)$. Note that this transformation defines our frame of reference. Eq. (1), therefore, in terms of the new variables is

$$D\left(\frac{\partial^2 C_\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial C_\alpha}{\partial r} + \frac{\partial^2 C_\alpha}{\partial x_1^2}\right) = \frac{\partial C_\alpha}{\partial t_1} + \left(u_0 F(r, t; a) - x_0'(t)\right)\frac{\partial C_\alpha}{\partial x_1} \quad (2)$$

We solve for the mean concentration profile across the axisymmetric pipe in the regime where the time taken for appreciable convective effects is large compared to diffusion effects along the r-axis. For a given t, if L is the spread of the solute, then this is valid when $\frac{a^2}{D}\frac{u_0}{L} = O(\epsilon)$, where ϵ is a small parameter. The spread can be taken to be the width of the distribution at half the maximum height.

Instead of $V(r,t;a)$ in Eq. (1) we will use $V(\frac{a_0 r}{a},t;a)$ which gives the correct result when $a_0 = a$, that is $V(r,t;a) = V(a_0\frac{r}{a},t;a)|_{a_0=a}$. We consider the limit $\epsilon \to 0$ where $u_0 = \frac{\tilde{u}_0}{\epsilon}$, $a = \epsilon a_0$, $z = \frac{r}{a} = \frac{r}{\epsilon a_0}$, and $x_0'(t) = \frac{\tilde{x}_0'(t)}{\epsilon}$, where $\tilde{u}_0$, a_0 and $\tilde{x}_0'$ are assumed fixed as $\epsilon \to 0$. Thus typical velocities are assumed to be $O(\frac{1}{\epsilon})$ and the radius of the pipe is assumed to be $O(\epsilon)$. Then define $C_\beta(z,x_1,t_1) = C_\alpha(r,x_1,t_1)$ at corresponding points, that is $C_\beta(\frac{r}{\epsilon a_0},x_1,t_1) = C_\alpha(r,x_1,t_1)$. Substituting the new expressions for the variables u_0 and r and simplifying, we get

$$\frac{D}{a_0^2}\left(\frac{\partial^2 C_\beta}{\partial z^2} + \frac{1}{z}\frac{\partial C_\beta}{\partial z}\right) + \epsilon^2 D\frac{\partial^2 C_\beta}{\partial x_1^2} = \epsilon^2\frac{\partial C_\beta}{\partial t_1} + \epsilon(\tilde{u}_0 F(a_0 z,t_1;a) - \tilde{x}_0'(t_1))\frac{\partial C_\beta}{\partial x_1} \tag{3}$$

Let $C_\beta(z,x_1,t_1) = C_0(z,x_1,t_1) + \epsilon C_1(z,x_1,t_1) + \epsilon^2 C_2(z,x_1,t_1) + O(\epsilon^3)$. Substituting this expression for C_β in Eq. (3) yields

$$\begin{aligned}\frac{D}{a_0^2}&\left(\frac{\partial^2 C_0}{\partial z^2} + \frac{1}{z}\frac{\partial C_0}{\partial z} + \epsilon\left(\frac{\partial^2 C_1}{\partial z^2} + \frac{1}{z}\frac{\partial C_1}{\partial z}\right)\right.\\ &\left.+\epsilon^2\left(\frac{\partial^2 C_2}{\partial z^2} + \frac{1}{z}\frac{\partial C_2}{\partial z}\right)\right) + \epsilon^2 D\frac{\partial^2 C_0}{\partial x_1^2}\\ &= \epsilon^2\frac{\partial C_0}{\partial t_1} + \epsilon(\tilde{u}_0 F(a_0 z,t_1;a) - \tilde{x}_0'(t_1))\frac{\partial C_0}{\partial x_1}\\ &\quad + \epsilon^2(\tilde{u}_0 F(a_0 z,t_1;a) - \tilde{x}_0'(t_1))\frac{\partial C_1}{\partial x_1}\end{aligned} \tag{4}$$

O(1) terms are

$$\frac{D}{a_0^2}\left(\frac{\partial^2 C_0}{\partial z^2} + \frac{1}{z}\frac{\partial C_0}{\partial z}\right) = 0 \tag{5}$$

The only solutions of 5 are linear combinations of $\log(z)$ and constants in z. Since $\log(z)$ is singular at zero, the only physical solutions are therefore independent of z. From now on, we shall write $C_0(x_1,t_1)$ since C_0 is independent of z.

The $O(\epsilon)$ terms are

$$\frac{D}{a_0^2}\left(\frac{\partial^2 C_1}{\partial z^2} + \frac{1}{z}\frac{\partial C_1}{\partial z}\right) = (\tilde{u}_0 F(a_0 z,t_1;a) - \tilde{x}_0'(t_1))\frac{\partial C_0}{\partial x_1} \tag{6}$$

Multiplying by z and rearranging, we get

$$\frac{D}{a_0^2}\frac{\partial}{\partial z}\left(z\frac{\partial C_1}{\partial z}\right) = z(\tilde{u}_0 F(a_0 z,t_1;a) - \tilde{x}_0'(t_1))\frac{\partial C_0}{\partial x_1} \tag{7}$$

Integrating and using the no-flux condition at the wall of the pipe, $(\frac{\partial C_\beta}{\partial z})_{z=1} = 0$, we get the following condition on $\tilde{x}_0'(t_1)$:

$$\tilde{x}_0'(t_1;\tilde{u}_0) = 2\int_0^1 \tilde{u}_0 z F(a_0 z,t_1;a)\,dz \tag{8}$$

The dependence on $\tilde{u}_0$ is made explicit for future reference. Recall that $u_0 = \frac{\tilde{u}_0}{\epsilon}$ and $x_0' = \frac{\tilde{x}_0'}{\epsilon}$. Eq. (8) shows the consistency of assuming that x_0' and u_0 are both the same order of magnitude, $O(\frac{1}{\epsilon})$. Now, we look for a solution of

$$\frac{D}{a_0^2}\left(\frac{\partial^2 C_1}{\partial z^2} + \frac{1}{z}\frac{\partial C_1}{\partial z}\right) = \left(\tilde{u}_0 F(a_0 z,t_1;a) - \tilde{x}_0'(t_1;\tilde{u}_0)\right)\frac{\partial C_0}{\partial x_1}$$
$$\left(\frac{\partial C_1}{\partial z}\right)_{z=1} = 0 \tag{9}$$

Integrating the first equation with respect to z and simplifying, we get

$$\frac{D}{a_0^2}\frac{\partial C_1}{\partial z} = \left(\frac{\tilde{u}_0}{z}\int_0^z \bar{z}F(a_0\bar{z},t_1;a)\,d\bar{z} - \frac{z}{2}\tilde{x}_0'(t_1;\tilde{u}_0)\right)\frac{\partial C_0}{\partial x_1} + \frac{\text{constant}}{z} \tag{10}$$

Integrating again with respect to z and simplifying, we get

$$C_1 = \frac{a_0^2}{D}\left(\int_0^z \frac{\tilde{u}_0}{\bar{z}}\int_0^{\bar{z}} \tilde{z}F(a_0\tilde{z},t_1;a)\,d\tilde{z}d\bar{z} - \frac{z^2}{4}\tilde{x}_0'(t_1;\tilde{u}_0)\right)\frac{\partial C_0}{\partial x_1} + \text{constant}\log(z) \tag{11}$$

We eliminate the $\log(z)$ term because of its singularity at $z = 0$, and we get the following expression for the solution C_1:

$$C_1 = \frac{a_0^2}{D}\left(\int_0^z \frac{\tilde{u}_0}{\bar{z}}\int_0^{\bar{z}} \tilde{z}F(a_0\tilde{z},t_1;a)\,d\tilde{z}d\bar{z} - \frac{z^2}{4}\tilde{x}_0'(t_1;\tilde{u}_0)\right)\frac{\partial C_0}{\partial x_1} \tag{12}$$

Finally, consider the $O(\epsilon^2)$ terms:

$$\frac{D}{a_0^2}\left(\frac{\partial^2 C_2}{\partial z^2} + \frac{1}{z}\frac{\partial C_2}{\partial z}\right) = \frac{\partial C_0}{\partial t_1} + (\tilde{u}_0 F(a_0 z,t_1;a) - \tilde{x}_0'(t_1;\tilde{u}_0))\frac{\partial C_1}{\partial x_1} - D\frac{\partial^2 C_0}{\partial x_1^2} \tag{13}$$

Differentiating Eq. (12) with respect to x_1, we get

$$\frac{\partial C_1}{\partial x_1} = \frac{a_0^2}{D}\left(\int_0^z \frac{\tilde{u}_0}{\bar{z}}\int_0^{\bar{z}} \tilde{z}F(a_0\tilde{z},t_1;a)\,d\tilde{z}d\bar{z} - \frac{z^2}{4}\tilde{x}_0'(t_1;\tilde{u}_0)\right)\frac{\partial^2 C_0}{\partial x_1^2} \tag{14}$$

and therefore

$$\frac{D}{a_0^2}\left(\frac{\partial}{\partial z}\left(z\frac{\partial C_2}{\partial z}\right)\right) = z\frac{\partial C_0}{\partial t} + z(\tilde{u}_0 F(a_0 z, t_1; a) - \tilde{x}_0'(t_1;\tilde{u}_0)) \times \frac{a_0^2}{D}\left(\int_0^z \frac{\tilde{u}_0}{\bar{z}}\int_0^{\bar{z}} \tilde{z}F(a_0\tilde{z}, t_1; a)\,\mathrm{d}\tilde{z}\mathrm{d}\bar{z} - \frac{z^2}{4}\tilde{x}_0'(t_1;\tilde{u}_0)\right) \times \frac{\partial^2 C_0}{\partial x_1^2} - zD\frac{\partial^2 C_0}{\partial x_1^2} \quad (15)$$

Integrating from 0 to 1 and simplifying yields

$$\frac{\partial C_0}{\partial t} = \left(2\frac{a_0^2}{D}\left(\int_0^1\left(\tilde{x}_0'(t_1;\tilde{u}_0)\tilde{u}_0 z\int_0^z \frac{1}{\bar{z}}\int_0^{\bar{z}}\tilde{z}F(a_0\tilde{z}, t_1; a)\,\mathrm{d}\tilde{z}\mathrm{d}\bar{z} + \frac{z^3}{4}\tilde{u}_0\tilde{x}_0'(t_1;\tilde{u}_0)F(a_0 z, t_1; a) - \tilde{u}_0 zF(a_0 z, t_1; a)\tilde{u}_0\int_0^z \frac{1}{\bar{z}}\int_0^{\bar{z}}\tilde{z}F(a_0\tilde{z}, t_1; a)\,\mathrm{d}\tilde{z}\mathrm{d}\bar{z} - \frac{1}{16}\tilde{x}_0'(t_1;\tilde{u}_0)^2\right)\mathrm{d}z\right) + D\right)\frac{\partial^2 C_0}{\partial x_1^2} \quad (16)$$

Since $a_0\tilde{u}_0 = au_0$, expressing F in terms of V, that is using $V(\frac{a_0}{a}r, t; a) = u_0(t)F(\frac{a_0}{a}r, t; a)$, and evaluating V at $a_0 = a$, we get as a final expression

$$\frac{\partial C_0}{\partial t} = \left(\frac{2a^2}{D}\left(\int_0^1\left(x_0'(t_1;u_0)z\int_0^z \frac{1}{\bar{z}}\int_0^{\bar{z}}\tilde{z}V(a\tilde{z}, t_1; a)\,\mathrm{d}\tilde{z}\mathrm{d}\bar{z} + \frac{z^3}{4}x_0'(t_1;u_0)V(az, t_1; a) - zV(az, t_1, a)\int_0^z \frac{1}{\bar{z}}\int_0^{\bar{z}}\tilde{z}V(a\tilde{z}, t_1; a)\,\mathrm{d}\tilde{z}\mathrm{d}\bar{z} - \frac{1}{16}x_0'(t_1;u_0)^2\right)\mathrm{d}z\right) + D\right)\frac{\partial^2 C_0}{\partial x_1^2} \quad (17)$$

This is the general Taylor diffusion equation for time-dependent flow. In the steady state, $V(r, t; a) = u_0(1 - (\frac{r}{a})^2)$. For $z = \frac{r}{a} = \frac{r}{\epsilon a_0}$, $V(az, t; a) = u_0(1 - z^2)$. Eq. (17) then simplifies to the time-independent, classical Taylor result. This is shown as follows: substituting $V(az, t; a) = u_0\,(1 - z^2)$ we get

$$\int_0^z \frac{u_0}{\bar{z}}\int_0^{\bar{z}}\tilde{z}(1-\tilde{z}^2)\,\mathrm{d}\tilde{z}\mathrm{d}\bar{z} = u_0\int_0^z\left(\frac{\bar{z}}{2} - \frac{\bar{z}^3}{4}\right)\mathrm{d}\bar{z} = \frac{u_0}{16}(4z^2 - z^4) \quad (18)$$

Substituting $F(az, t; a) = 1 - z^2$ in Eq. (8) and simplifying we get the following expression for $x_0'(t; u_0)$:

$$x_0'(t;u_0) = 2u_0\int_0^1 z(1-z^2)\,\mathrm{d}z = 2u_0\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{u_0}{2} \quad (19)$$

So $x_0'(t; u_0)$ is the average speed over the cross section of the pipe. It follows that the integral in Eq. (17) simplifies to

$$\int_0^1\left(\frac{u_0^2}{2(16)}z(4z^2 - z^4) + \frac{z^3}{4}\frac{u_0^2}{2}(1-z^2) - \frac{u_0^2}{16}z(1-z^2)(4z^2-z^4) - \frac{u_0^2}{4}\frac{1}{16}\right)\mathrm{d}z = \frac{u_0^2}{2}\left(\frac{1}{16}\left(1-\frac{1}{6}\right) + \left(\frac{1}{16} - \frac{1}{24}\right) - \frac{1}{32} - \frac{2}{16}\left(1 - \frac{5}{6} + \frac{1}{8}\right)\right) = \frac{u_0^2}{2}\frac{1}{192} \quad (20)$$

Eq. (17) therefore reduces to

$$\frac{\partial C_0}{\partial t} = \left(a^2\frac{u_0^2}{192D} + D\right)\frac{\partial^2 C_0}{\partial x_1^2} \quad (21)$$

which is the classical result that Taylor obtained in his 1953 paper. Note that the effective diffusion coefficient has two terms. One is just the molecular diffusion D. The other $\frac{a^2 u_0^2}{192D}$, remarkably depends inversely on D. The effective diffusion coefficient in our result, Eq. (17), although much more complicated, has this same dependence on D, in that one term is D itself, and the rest of the expression has a factor of $\frac{1}{D}$.

3. Application

We solved for the concentration of a solute which is present at time $t = 0$ with a given distribution $C(r, x, 0)$ in an axisymmetric pipe in which the pressure gradient driving the flow is given by $\frac{\partial p}{\partial x} = A\sin(w_0 t) + B$. We then solved for the axisymmetric flow along the pipe according to the equation

$$\rho\frac{\partial V}{\partial t} + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right) \quad (22)$$

together with the no slip condition at the boundary $V|_a = 0$ and the condition that the flow is finite at $r = 0$. Substituting $\frac{\partial p}{\partial x} = A\sin(w_0 t) + B$ into the equation and solving for V using Fourier analysis, we get

$$V(r, t; a) = \frac{-A}{2w_0\rho}\left(-1 + \frac{J_0\left(-(-1)^{\frac{3}{4}}r\sqrt{-w_0\frac{\rho}{\mu}}\right)}{J_0\left(-(-1)^{\frac{3}{4}}a\sqrt{-w_0\frac{\rho}{\mu}}\right)}\right)\mathrm{e}^{-\mathrm{i}w_0 t} + \frac{-A}{2w_0\rho}\left(-1 + \frac{J_0\left(-(-1)^{\frac{3}{4}}r\sqrt{w_0\frac{\rho}{\mu}}\right)}{J_0\left(-(-1)^{\frac{3}{4}}a\sqrt{w_0\frac{\rho}{\mu}}\right)}\right)\mathrm{e}^{\mathrm{i}w_0 t} + \frac{B}{4\mu}(r^2 - a^2) \quad (23)$$

where J_0 is the zeroth order Bessel function of the first kind. To be able to use the generalized Taylor Eq. (17) we numerically calculated the effective diffusion coefficient (a function of time) and then used a heat equation solver to solve for the concentration at a given time t along the pipe. For comparison, we then solved the corresponding convection–diffusion equation using FEMLAB 3.0 (Comsol, MA). We used the "Transient Convection and Diffusion" FEMLAB 3.0 module with parameters $A = 55$, $B = -40$, $\rho = 1$, $\mu = 1$, $a = 0.25$, $w_0 = \frac{2\pi}{T}$ and $T = 0.75$. The units we used were centimeters for distance, seconds for time and Pascals for pressure. Note that the amplitude of the oscillation A in the

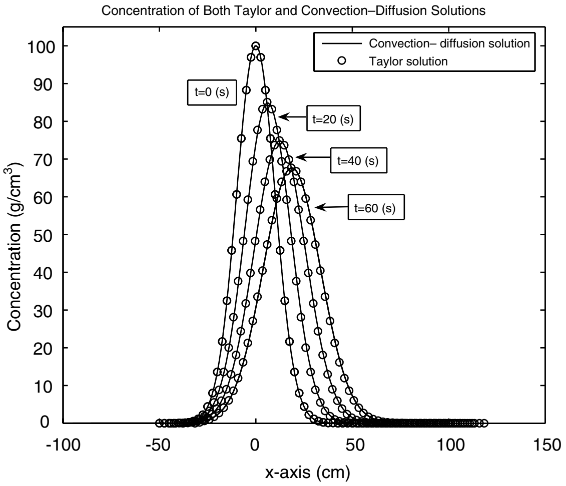

Fig. 1. Comparison of convection–diffusion solution with Taylor solution.

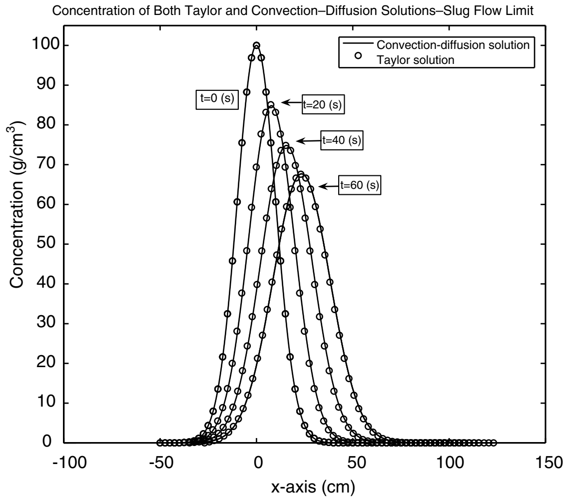

Fig. 2. Comparison of convection–diffusion solution with Taylor solution in the slug flow limit.

pressure gradient is comparable to the amplitude B of the mean pressure gradient. As initial condition, we used $C(r,x,0) = f(x) = 100e^{-0.005x^2}$.

Fig. 1 is a comparison of the generalized Taylor diffusion equation solution and the axisymmetric convection–diffusion solution at $t = 0$, 20, 40, 60. Note that many cycles of the oscillatory flow have elapsed between each of the curves shown in the figure. Fig. 1 shows that there is excellent agreement between the two sets of solutions.

We also consider the limit of high Womersley number $\alpha = r\sqrt{\frac{\omega}{\nu}}$, where $\nu = \frac{\mu}{\rho}$, and low Womersley number for the flow produced with the sinusodial pressure gradient given by $\frac{\partial p}{\partial x} = A\sin(\omega t) + B$. For Womersley number sufficiently greater than one, the velocity profile at a given location x along the pipe is flat near the center, with a Stokes layer near the wall. This also known as the slug flow limit. For Womersley number sufficiently less than one, the flow is quasi-steady, and the velocity profile at a given x location along the pipe is approximately parabolic. This is also known as the slow varying Hagen–Poiseuille limit. We fix the parameters $A = 55$, $B = -2$, $\rho = 1$, $\mu = 0.04$, $a = 0.25$, $T = 0.75$, vary the frequency ω, and use the same initial value $C(r,x,0) = f(x) = 100e^{-0.005x^2}$. We let $\omega = 2\pi * \frac{5}{T}$ for the high frequency limit or the slug flow limit, giving a Womersley number $\alpha = 8.09$. We let $\omega = 2\pi * \frac{1}{T*100\,000}$ for the low frequency limit or the slow varying Hagen–Poiseuille limit, giving a Womersley number $\alpha = 0.011$. Figs. 2 and 3 correspond to the slug flow limit, and the slow varying Hagen–Poiseuille limit respectively. Both figures show excellent agreement between the convection–diffusion solution and the Taylor solution based on Eq. (17). The parameters were chosen to be physiologically relevant. The period of the cardiac cycle is taken to be 0.75 s, and blood is taken to have a kinematic viscosity $\nu = \frac{\mu}{\rho} = 0.04\,\frac{\text{cm}^2}{s}$. The

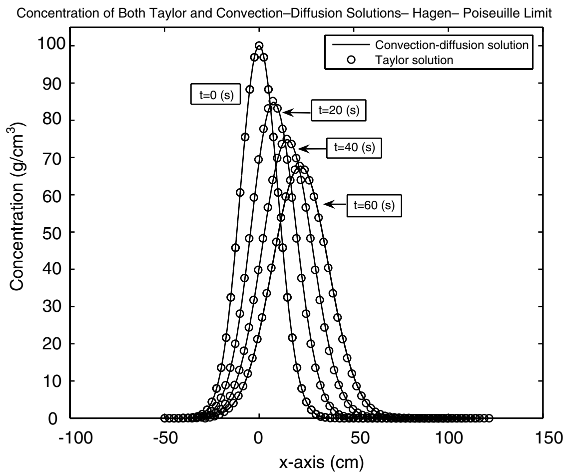

Fig. 3. Comparison of convection–diffusion solution with Taylor solution in the Hagen–Poiseuille limit.

radius a of the common carotid artery in a human is approximately 0.25 cm. The pressure gradient parameters were chosen to allow for reversal of flow near the artery walls in the slug flow limit, which is typical of flow in the arteries. The Womersley number in the circulation varies greatly and can go up in the large arteries to approximately the upper limit which we have chosen for the slug flow, to as low in the small arteries as the lower limit we have chosen for the Hagen–Poiseuille flow.

4. Conclusion

This paper generalizes the 1953 Taylor diffusion result to time-dependent flows. To verify the theory, we have shown that the axial concentration profile calculated by means of the axisymmetric convection–diffusion equation and by means of the one-dimensional generalized Taylor diffusion equation are in good agreement.

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